Elektriciteit & Magnetisme 2, 28-10-2008

1a) $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} = \mathcal{E}$

$\Phi = \int \vec{B}\cdot d\vec{a} = BLvt + \text{constante}$

$\mathcal{E} = -\frac{d\phi}{dt} = -BLv$

$I = \mathcal{E}/R = -\frac{BLv}{R}$, richting: tegen de klok in

b) $F = ILB = -\frac{B^2L^2v}{R}$, richting: ~~[illegible]~~ naar links

c) $F = -\frac{B^2L^2v}{R} = m\frac{dv}{dt}$

$v = v_0 e^{-\frac{B^2L^2}{Rm}t}$

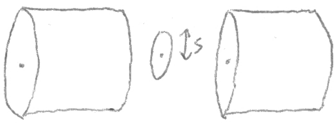
d) $W = \int F\,dx = \int -\frac{B^2L^2v}{R}\,d(vt) = -\frac{B^2L^2}{R}\int v^2\,dt = -\frac{B^2L^2}{R}\int_0^{\infty} v_0^2 e^{-\frac{2B^2L^2}{Rm}t}\,dt$

$= -\frac{B^2L^2}{R}v_0^2\left[-\frac{Rm}{2B^2L^2}e^{-\frac{2B^2L^2}{Rm}t}\right]_{t=0}^{\infty} = \frac{1}{2}mv_0^2$

2a) $\sigma = \frac{Q}{\pi a^2}$

$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z} = \frac{Q}{\epsilon_0\pi a^2}\hat{z} = \frac{It}{\epsilon_0\pi a^2}\hat{z}$

b)



Flux door Ampèrian Loop: $\Phi_E = \frac{It}{\epsilon_0\pi a^2}\pi s^2 = \frac{Its^2}{\epsilon_0 a^2}$

$\oint \vec{B}\cdot d\vec{l} = 2\pi sB = \mu_0 I_{enc} + \mu_0\epsilon_0\frac{d\Phi_E}{dt} = \frac{\mu_0 I s^2}{a^2}$

$\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}$

c) $u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\left(\frac{I^2t^2}{\epsilon_0\pi^2a^4} + \frac{\mu_0 I^2 s^2}{4\pi^2 a^4}\right)$

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{It}{\epsilon_0\pi a^2}\frac{Is}{2\pi a^2}(\hat{z}\times\hat{\phi}) = -\frac{I^2ts}{2\epsilon_0\pi^2a^4}\hat{s}$

$\frac{\partial}{\partial t}u_{em} = \frac{I^2t}{\epsilon_0\pi^2a^4}$

$-\vec{\nabla}\cdot\vec{S} = \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{I^2ts^2}{2\epsilon_0\pi^2a^4}\right) = \frac{I^2t}{\epsilon_0\pi^2a^4} = \frac{\partial}{\partial t}u_{em}$

d) $E = \int u_{em}\, d\tau = \int_0^w \int_0^{2\pi} \int_0^a \frac{1}{2}\left(\frac{I^2 t^2}{\varepsilon_0 \pi^2 a^4} + \frac{\mu_0 I^2 s^2}{4\pi^2 a^4} \right) s\, ds\, d\phi\, dz$

$= 2\pi w \frac{1}{2}\left(\frac{I^2 t^2}{\varepsilon_0 \pi^2 a^4} \frac{1}{2} a^2 + \frac{\mu_0 I^2}{4\pi^2 a^4} \frac{1}{4} a^4 \right) = w\left(\frac{I^2 t^2}{2\varepsilon_0 \pi a^2} + \frac{\mu_0 I^2}{16\pi} \right)$

$\oint \vec{S} \cdot d\vec{a} = \int_0^{2\pi} \int_0^w - \frac{I^2 t a}{2\varepsilon_0 \pi^2 a^4}\, a\, dz\, d\phi = -2\pi w \frac{I^2 t}{2\varepsilon_0 \pi^2 a^2} = -w \frac{I^2 t}{\cancel{2}\varepsilon_0 \pi a^2}$

$\frac{dE}{dt} = w \frac{I^2 t}{\varepsilon_0 \pi a^2} = -\oint \vec{S} \cdot d\vec{a}$

3a) $t' = \gamma(t - \frac{v}{c^2} x)$ $\quad x' = \gamma(x - vt)$ $\quad y' = y$ $\quad z' = z$

$V' = \gamma(V - vA_x)$ $\quad A_x' = \gamma(A_x - \frac{v}{c^2} V)$ $\quad A_y' = A_y$ $\quad A_z' = A_z$

b) $V' = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r'}$

$\vec{A}' = \vec{0}$

Transformatie naar S:

$V' = \gamma(V - vA_x) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r'}$ $\quad (*)$

$A_x' = \gamma(A_x - \frac{v}{c^2} V) = 0$ $\quad (**)$

$A_y = A_z = 0$

Uit $(**)$ volgt $A_x = \frac{v}{c^2} V$, invullen in $(*)$:

$\gamma(V - \frac{v^2}{c^2} V) = V\gamma(1 - \frac{v^2}{c^2}) = V\gamma^{-1} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r'} \Rightarrow V = \frac{\gamma}{4\pi\varepsilon_0} \frac{Q}{r'}$

$A_x = \frac{v}{c^2} V = \frac{v}{c^2} \frac{\gamma}{4\pi\varepsilon_0} \frac{Q}{r'}$

$r' = \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{\gamma^2 (x - vt)^2 + y^2 + z^2}$

$V = \frac{\gamma}{4\pi\varepsilon_0} \frac{Q}{\sqrt{\gamma^2 (x - vt)^2 + y^2 + z^2}}$

$\vec{A} = \frac{v}{c^2} \frac{\gamma}{4\pi\varepsilon_0} \frac{Q}{\sqrt{\gamma^2 (x - vt)^2 + y^2 + z^2}} = \frac{1}{c^2} \vec{v}\, V$

c) $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

d) $\dfrac{\partial V}{\partial t} = \dfrac{\gamma Q}{4\pi\varepsilon_0} \dfrac{-\gamma^2 v(x-vt)}{\sqrt{\gamma^2(x-vt)^2+y^2+z^2}}$

$\vec{\nabla}\cdot\vec{A} = \dfrac{v}{c^2} \dfrac{\gamma Q}{4\pi\varepsilon_0} \dfrac{\gamma^2(x-vt)}{\sqrt{\gamma^2(x-vt)^2+y^2+z^2}} = -\dfrac{1}{c^2}\dfrac{\partial V}{\partial t}$

4a) $\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

$\vec{\nabla}\cdot\vec{E} = \dfrac{\partial}{\partial y} E_0 \cos(kx-\omega t) = 0$

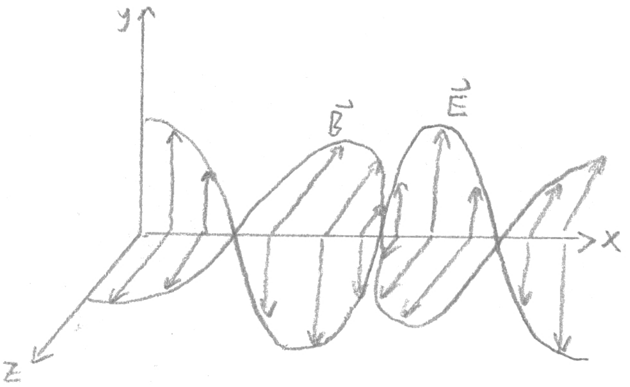
$\vec{\nabla}\times\vec{E} = \hat{z}\dfrac{\partial}{\partial x} E_0\cos(kx-\omega t) = -E_0 k\sin(kx-\omega t)\hat{z} = -\dfrac{\partial\vec{B}}{\partial t}$

$\vec{B} = E_0\dfrac{k}{\omega}\cos(kx-\omega t)\hat{z} = \dfrac{1}{c}E_0\cos(kx-\omega t)\hat{z}$

$\vec{\nabla}\cdot\vec{B} = \dfrac{\partial}{\partial z}\dfrac{1}{c}E_0\cos(kx-\omega t) = 0$

$\vec{\nabla}\times\vec{B} = -\hat{y}\dfrac{\partial}{\partial x}\dfrac{1}{c}E_0\cos(kx-\omega t) = \dfrac{k}{c}E_0\sin(kx-\omega t)\hat{y}$

$\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} = \dfrac{\omega}{c^2}E_0\sin(kx-\omega t)\hat{y} = \dfrac{k}{c}E_0\sin(kx-\omega t)\hat{y} = \vec{\nabla}\times\vec{B}$



b) $\vec{S} = \dfrac{1}{\mu_0}(\vec{E}\times\vec{B}) = \dfrac{1}{\mu_0 c}E_0^2\cos^2(kx-\omega t)\hat{x}$

$\langle\vec{S}\rangle = \dfrac{1}{2\mu_0 c}E_0^2\hat{z} = \vec{I}$

c) $\vec{E}' = \gamma(E_y - vB_z)\hat{y} = \gamma E_0(1-\frac{v}{c})\cos(kx-\omega t)$

$\vec{B}' = \gamma(B_z - \frac{v}{c^2}E_y)\hat{z} = \gamma\frac{1}{c}E_0(1-\frac{v}{c})\cos(kx-\omega t)$

$kx-\omega t = \gamma(k(x'+vt') - \omega(t'+\frac{v}{c^2}x')) = \gamma((k-\omega\frac{v}{c^2})x' + (kv-\omega)t') = k'x'-\omega't'$

$\vec{E}' = E_0\gamma(1-\frac{v}{c})\cos(k'x'-\omega't')$

$\vec{B}' = \frac{1}{c}E_0\gamma(1-\frac{v}{c})\cos(k'x'-\omega't')$

d) $\omega' = \gamma(\omega-kv) = \gamma\omega(1-\frac{v}{c}) = \omega(1-\frac{v^2}{c^2})^{-1/2}(1-\frac{v}{c}) = \omega(1-\frac{v}{c})^{-1/2}(1+\frac{v}{c})^{-1/2}(1-\frac{v}{c}) = \omega\sqrt{\dfrac{1-v/c}{1+v/c}}$

$k' = \gamma(k-\omega\frac{v}{c^2}) = k\sqrt{\dfrac{1-v/c}{1+v/c}} \Rightarrow \lambda' = \dfrac{2\pi}{k'} = \dfrac{2\pi}{k}\sqrt{\dfrac{1+v/c}{1-v/c}}$

$v' = \lambda'\nu' = \lambda'\dfrac{\omega'}{2\pi} = \dfrac{2\pi}{k}\dfrac{\omega}{2\pi}\sqrt{\dfrac{1-v/c}{1+v/c}}\sqrt{\dfrac{1+v/c}{1-v/c}} = \dfrac{\omega}{k} = c$

$\lim\limits_{v\to c}\omega' = \lim\limits_{v\to c}\omega\sqrt{\dfrac{1-v/c}{1+v/c}} = 0$

$\lim\limits_{v\to c} E_0\sqrt{\dfrac{1-v/c}{1+v/c}} = 0$

$\lim\limits_{v\to c} I = \lim\limits_{v\to c}\dfrac{1}{2\mu_0 c}(E_0')^2 = 0$